

Decay Constants of Beauty Mesons from QCD Sum Rules

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Abstract. Our recently completed analysis of the decay constants of both pseudoscalar and vector beauty mesons reveals that in the bottom-quark sector two specific features of the sum-rule predictions show up: (i) For the input value of the bottom-quark mass in the $\overline{\text{MS}}$ scheme $\overline{m}_b(\overline{m}_b) \approx 4.18$ GeV, the sum-rule result $f_B \approx 210\text{--}220$ MeV for the B meson decay constant is substantially larger than the recent lattice-QCD finding $f_B \approx 190$ MeV. Requiring QCD sum rules to reproduce the lattice-QCD value of f_B yields a significantly larger b -quark mass: $\overline{m}_b(\overline{m}_b) = 4.247$ GeV. (ii) Whereas QCD sum-rule predictions for the charmed-meson decay constants f_D, f_{D_s}, f_{D^*} and $f_{D_s^*}$ are practically independent of the choice of renormalization scale, in the beauty sector the results for the decay constants—and especially for the ratio f_{B^*}/f_B —prove to be very sensitive to the specific scale setting.

1 Correlator, operator product expansion, and heavy-quark mass schemes

The starting point of our QCD sum-rule evaluation of the decay constants [1] of beauty mesons is the time-ordered product of two meson interpolating currents, *viz.*, $j_5(x) = (m_b + m) \bar{q}(x) i \gamma_5 b(x)$ for the B meson and $j_\mu(x) = \bar{q}(x) \gamma_\mu b(x)$ for the B^* meson. The correlator of pseudoscalar currents is defined by

$$\Pi(p^2) \equiv i \int d^4x e^{i p x} \left\langle 0 \left| T(j_5(x) j_5^\dagger(0)) \right| 0 \right\rangle.$$

The Borel transform of this correlation function depends on some Borel parameter τ and takes the form

$$\Pi(\tau) = f_B^2 M_B^4 \exp(-M_B^2 \tau) + \int_{(M_{B^*} + M_P)^2}^{\infty} ds e^{-s\tau} \rho_{\text{hadr}}(s) = \int_{(m_b + m)^2}^{\infty} ds e^{-s\tau} \rho_{\text{pert}}(s, \mu) + \Pi_{\text{power}}(\tau, \mu).$$

The B -meson decay constant f_B is defined by $\langle 0 | j_5(0) | B \rangle = f_B M_B^2$. In order to remove all excited-state contributions, we adopt the standard assumption of quark–hadron duality: the contributions of excited

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states are compensated by the perturbative contribution above an *effective continuum threshold* $s_{\text{eff}}(\tau)$ which *differs* from the physical continuum threshold. Applying this Ansatz leads, for the B meson, to

$$f_B^2 M_B^4 \exp(-M_B^2 \tau) = \int_{(m_b+m)^2}^{s_{\text{eff}}(\tau)} ds e^{-s\tau} \rho_{\text{pert}}(s, \mu) + \Pi_{\text{power}}(\tau, \mu) \equiv \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)).$$

The right-hand side of the above relation constitutes what we call the *dual correlator* $\Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau))$. The best-known three-loop calculation of the perturbative spectral density ρ_{pert} has been performed as an expansion in terms of the $\overline{\text{MS}}$ strong coupling, $a(\mu) \equiv \alpha_s(\mu)/\pi$ and the b -quark pole mass M_b [2, 3]:

$$\rho_{\text{pert}}(s, \mu) = \rho^{(0)}(s, M_b^2) + a(\mu) \rho^{(1)}(s, M_b^2) + a^2(\mu) \rho^{(2)}(s, M_b^2, \mu) + \dots$$

An tantalizing feature of the pole-mass operator product expansion (OPE) is that each of the known perturbative contributions to the dual correlator is positive. Unfortunately, such a pole-mass OPE does not provide a visible hierarchy of the perturbative contributions and thus poses strong doubts that the $O(\alpha_s^2)$ -truncated pole-mass OPE can provide reliable estimates of the decay constants. An alternative [4] is to reorganize the perturbative expansion in terms of the running $\overline{\text{MS}}$ mass, $\overline{m}_b(\nu)$, by substituting in the spectral densities $\rho^{(i)}(s, M_b^2) M_b$ by its perturbative expansion in terms of the running mass $\overline{m}_b(\nu)$

$$M_b = \overline{m}_b(\nu) [1 + a(\nu) r_1 + a^2(\nu) r_2 + \dots].$$

The spectral densities in the $\overline{\text{MS}}$ scheme are obtained by expanding the pole-mass spectral densities in powers of $a(\mu)$ and omitting terms of order $O(a^3)$ and higher; starting with order $O(a)$ they contain two parts: the result of Refs. [2, 3] and the impact of lower perturbative orders arising upon expansion of the pole mass in terms of the running mass. In this way, because of the truncation of the perturbative series, one gets an explicit *unphysical* dependence of both dual correlator and extracted decay constant on the scale μ . In principle, any scale should be equally good. In practice, however, the hierarchy of perturbative contributions to the dual correlator depends on the precise choice of the scale, opening the possibility to choose the scale μ such that the hierarchy of the new perturbative expansion is improved.

Let us define the scale $\tilde{\mu}$ by demanding $M_b = \overline{m}_b(\tilde{\mu})$. Using the $O(a^2)$ relation between running and pole masses, we obtain numerically $\tilde{\mu} \approx 2.23$ GeV. The perturbative hierarchy of the $\overline{\text{MS}}$ expansion at this scale is *worse* than that of the pole-mass expansion: The $O(1)$ spectral densities coincide, whereas the $O(a)$ spectral density in the $\overline{\text{MS}}$ scheme receives a positive contribution compared to the pole-mass scheme. For smaller scales, *i.e.*, $\mu < \tilde{\mu}$, the hierarchy of the $\overline{\text{MS}}$ expansion becomes even worse with decreasing μ . For $\mu > \tilde{\mu}$, first the hierarchy of the $\overline{\text{MS}}$ expansion improves for increasing μ , see Fig. 1. However, as soon as the scale μ becomes sufficiently larger than $\tilde{\mu}$, the “induced” contributions, which primarily reflect the poor behaviour of the expansion of the pole mass in terms of the running mass, overtake the “genuine” contributions. This is illustrated by Fig. 1: at $\mu = 4$ GeV, the $O(1)$ contribution to the dual correlator rises steeply with τ , whereas the $O(a)$ contribution becomes negative in order to compensate the rise of the $O(1)$ contribution. Finally, for large values of μ we observe a compensation between the “induced” contributions. One may expect that, in this case, the accuracy of the expansion will deteriorate. This is reflected by a strong scale dependence of the extracted values of f_B and f_{B^*} : as μ rises, the $O(a)$ term “undercompensates” the rise of the $O(1)$ term for the B meson and f_B increases with μ , whereas it overcompensates the rise of the $O(1)$ term for the B^* meson and f_{B^*} decreases with rising μ ; as a consequence, the ratio f_{B^*}/f_B proves to be particularly sensitive to the precise value of μ .

Now, returning to the pole-mass expansion for the b quark, we note that the hierarchy is not too bad and can be easily improved by switching to the running mass and choosing the scale μ slightly above $\tilde{\mu}$.

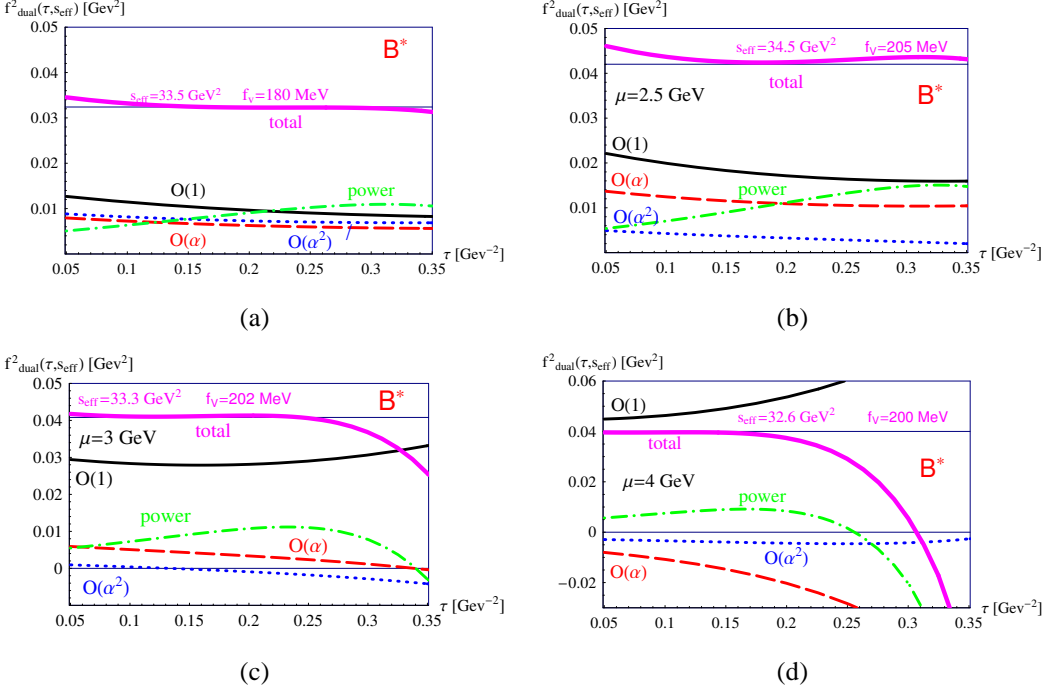


Figure 1. QCD sum-rule estimates for the B^* meson using either pole-mass or running-mass scheme at different scales: the $O(\alpha_s^2)$ -truncated pole-mass OPE shows no hierarchy of the perturbative expansion and cannot be used. Even the hierarchy of the running-mass OPE is not automatic but depends strongly on the renormalization scale μ . For $\bar{m}_b(\bar{m}_b) = 4.18$ GeV, the two-loop pole mass is $M_b = 4.80$ GeV. For each case, a constant effective threshold s_{eff} is determined by requiring maximal stability of the predictions in the Borel window $0.05 \leq \tau \text{ (GeV}^{-2}\text{)} \leq 0.15$. Bold (lilac) lines—total results, solid (black) lines— $O(1)$ contributions, dashed (red) lines— $O(\alpha_s)$ contributions, dotted (blue) lines— $O(\alpha_s^2)$ contributions, dot-dashed (green) lines—power contributions. Results from pole-mass OPE (a) versus running mass OPE for renormalization scale $\mu = 2.5$ GeV (b), $\mu = 3$ GeV (c), and $\mu = 4$ GeV (d).

Furthermore, we may, for instance, require that the $O(\alpha_s)$ contribution to the dual correlator remains positive in the working range of τ . (For a positive-definite dual correlator, it is not a strictly necessary but, for obvious reasons, a highly welcome feature if each of the perturbative contributions is positive.) In this case, we may expect to arrive at reliable results by setting $\mu = 2.5\text{--}3$ GeV for the B^* meson and $\mu = 2.5\text{--}3.5$ GeV for the B meson. An additional important argument in favour of such “low- μ ” choice is that the ratio f_{B^*}/f_B proves to be definitely less than unity at scales $\mu \approx m_b$, whereas it emerges close to unity for $\mu = 2.5\text{--}3$ GeV, in full agreement with heavy-quark expansion and hints from lattice QCD.

The results for the B^* -meson decay constant shown in Fig. 1 have been found by employing, for the b -quark mass, $\bar{m}_b(\bar{m}_b) = (4.18 \pm 0.030)$ GeV [5] and, for the other relevant OPE parameters, the values

$$\begin{aligned} \bar{m}_d(2 \text{ GeV}) &= (3.5 \pm 0.5) \text{ MeV} , & \bar{m}_s(2 \text{ GeV}) &= (95 \pm 5) \text{ MeV} , \\ \alpha_s(M_Z) &= 0.1184 \pm 0.0007 , & \langle \bar{q}q \rangle(2 \text{ GeV}) &= -[(269 \pm 17) \text{ MeV}]^3 , \\ \frac{\langle \bar{s}s \rangle(2 \text{ GeV})}{\langle \bar{q}q \rangle(2 \text{ GeV})} &= 0.8 \pm 0.3 , & \left\langle \frac{\alpha_s}{\pi} GG \right\rangle &= (0.024 \pm 0.012) \text{ GeV}^4 . \end{aligned}$$

To compare the results obtained from, on the one hand, the running-mass OPE and, on the other hand, the pole-mass OPE, we recalculate the pole mass from the $O(a^2)$ relation between \overline{m}_b and M_b , finding $M_b = 4.8$ GeV. Merely for illustrating the main features of the dual correlators, the sum-rule estimates shown in Fig. 1 are extracted for a τ -independent effective threshold $s_{\text{eff}} = \text{const}$. Its value in each case is deduced by requiring maximal stability of the extracted decay constant in the chosen Borel window.

2 Extraction of observables such as decay constants from QCD sum rules

The well-established procedures of QCD sum rules pave a straightforward path to extract observables:

1. Determine a reasonable Borel window, that is, an interval of the Borel parameter τ defined such that the OPE provides an accurate description of the exact correlator: higher-order radiative and power corrections have to be under control while, at the same time, the ground state contributes “sizeably” to the correlator; our τ window for, *e.g.*, the B meson reads $0.05 \lesssim \tau \text{ (GeV}^{-2}\text{)} \lesssim 0.18$.
2. Define and apply an appropriate criterion for fixing the effective continuum threshold $s_{\text{eff}}(\tau)$. To this end, we employ an earlier developed algorithm [6–9] that allows for a reliable extraction of the ground-state properties in quantum mechanics and of the charmed-meson decay constants in QCD. We introduce the *dual invariant mass* M_{dual} and the *dual decay constant* f_{dual} by defining

$$M_{\text{dual}}^2(\tau) \equiv -\frac{d}{d\tau} \log \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)), \quad f_{\text{dual}}^2(\tau) \equiv \frac{\exp(M_B^2 \tau)}{M_B^4} \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)).$$

The dual mass should reproduce the true ground-state mass M_B ; its deviation from M_B quantifies the contamination of $\Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau))$ by excited states. We determine the behaviour of $s_{\text{eff}}(\tau)$ by starting from a convenient Ansatz for $s_{\text{eff}}(\tau)$ and minimizing the deviation of the predicted M_{dual} from the observed M_B in the τ window by varying $s_{\text{eff}}(\tau)$. Since we need to know the behaviour of $s_{\text{eff}}(\tau)$ only in the limited τ window defined before, it suffices to consider merely polynomials in τ (which Ansatz allows, of course, also for the case of $s_{\text{eff}}(\tau)$ being a τ -independent constant):

$$s_{\text{eff}}(\tau) = \sum_{j=0}^n s_j^{(n)} \tau^j, \quad \chi^2 \equiv \frac{1}{N} \sum_{i=1}^N [M_{\text{dual}}^2(\tau_i) - M_B^2]^2.$$

3. With the variational result for $s_{\text{eff}}(\tau)$ at hand, $f_{\text{dual}}(\tau)$ yields the decay-constant estimate sought.

As all outcomes for hadron observables extracted from QCD sum rules do, also the predictions for decay constants are sensitive to the input values of all the parameters entering in one’s OPE—resulting in what we call, for clear reasons, their OPE-related uncertainty—and to the particularities of the route followed to get the effective threshold as a function of τ —contributing to their systematic uncertainty.

OPE-related uncertainty: By assuming a Gaussian distribution for the numerical value of each of the parameters required as OPE input except for the renormalization scales μ and ν , for which we assume uniform distributions in the range $3 \text{ GeV} \leq \mu, \nu \leq 6 \text{ GeV}$, we may estimate the size of the OPE-related uncertainty by performing a bootstrap analysis. The resulting distribution of decay constants proves to be close to Gaussian shape. So, the quoted OPE-related error should be understood as a Gaussian error.

Systematic uncertainty: The systematic uncertainty is an immediate consequence of the intrinsically limited accuracy of the QCD sum-rule approach and thus poses, without surprise, a delicate problem. Considering polynomial parameterizations of the effective continuum threshold $s_{\text{eff}}(\tau)$ for toy models within quantum mechanics, we could demonstrate that the band of results found from linear, quadratic, and cubic Ansätze for $s_{\text{eff}}(\tau)$ encompasses the true value of the decay constant. Thus, the half-width of this band should be regarded as a realistic estimate for the systematic uncertainty of such an extraction.

3 Decay constants of pseudoscalar and vector beauty mesons $B_{(s)}$ and $B_{(s)}^*$

The decay constants f_B and f_{B^*} emerging from application of our extraction procedure exhibit a strong sensitivity to the value used for $m_b \equiv \overline{m}_b(\overline{m}_b)$ (keeping fixed all other parameters relevant for the OPE):

- The decay constant $f_B^{\text{dual}}(m_b, \mu)$ of the pseudoscalar meson B behaves, as function of m_b and μ , like

$$f_B^{\text{dual}}(m_b, \mu = \mu^*) = 192.6 \text{ MeV} - 13 \text{ MeV} \left(\frac{m_b - 4.247 \text{ GeV}}{0.034 \text{ GeV}} \right), \quad \mu^* = 5.59 \text{ GeV},$$

$$f_B^{\text{dual}}(m_b = 4.247 \text{ GeV}, \mu) = 192.6 \text{ MeV} \left(1 - 0.0015 \log \frac{\mu}{\mu^*} + 0.030 \log^2 \frac{\mu}{\mu^*} + 0.061 \log^3 \frac{\mu}{\mu^*} \right).$$

- For the vector meson B^* , our *preliminary* results for the dual decay constant $f_{B^*}^{\text{dual}}(m_b, \mu)$ are given by

$$f_{B^*}^{\text{dual}}(m_b, \mu = \mu^*) = 186.4 \text{ MeV} - 10 \text{ MeV} \left(\frac{m_b - 4.247 \text{ GeV}}{0.034 \text{ GeV}} \right), \quad \mu^* = 5.82 \text{ GeV},$$

$$f_{B^*}^{\text{dual}}(m_b = 4.247 \text{ GeV}, \mu) = 186.4 \text{ MeV} \left(1 + 0.106 \log \frac{\mu}{\mu^*} + 0.337 \log^2 \frac{\mu}{\mu^*} + 0.173 \log^3 \frac{\mu}{\mu^*} \right).$$

The scale μ^* is obtained by averaging over our decay-constant predictions for B and B^* shown in Fig. 2.

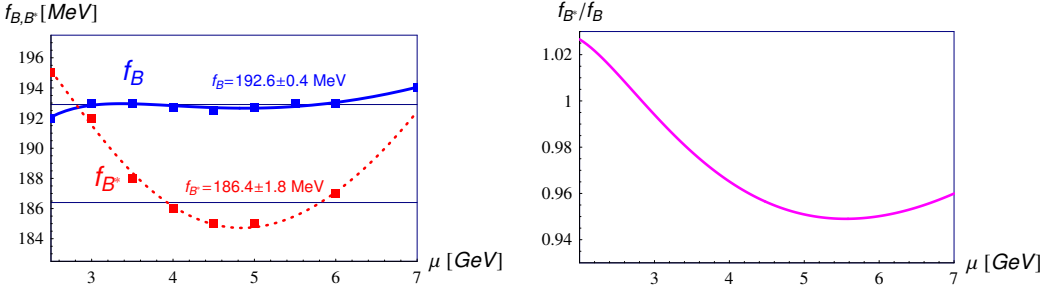


Figure 2. The μ dependence of the decay constants f_B and f_{B^*} (left) and the ratio f_{B^*}/f_B (right), corresponding to central values of all OPE parameters except for μ and to a quadratic Ansatz for the effective continuum threshold.

Our final sum-rule predictions for the decay constants under consideration depend to a large extent on the chosen input value of the b -quark mass m_b and on the way one deals with the dependence on μ :

- Assuming for f_B and f_{B^*} a flat distribution in the interval $\mu \in (3 \text{ GeV}, 6 \text{ GeV})$ and averaging over μ in this range clearly yields for their ratio $f_{B^*}/f_B < 1$, largely independent of the precise value of m_b .
- Using $m_b = 4.18 \text{ GeV}$ yields $f_B > 210 \text{ MeV}$, in clear tension with recent lattice-QCD results for f_B .

In Ref. [10], we noticed that requesting the sum-rule prediction for f_B to reproduce the lattice results requires the substantially higher b -quark mass $m_b = 4.247 \text{ GeV}$. Now, averaging for this m_b our results found for a *quadratic* Ansatz for the effective threshold over μ in the range $\mu \in (3 \text{ GeV}, 6 \text{ GeV})$ yields

$$f_B = (192.6 \pm 1.6) \text{ MeV},$$

$$f_{B^*} = (186.4 \pm 3.2) \text{ MeV},$$

$$f_{B_s} = (231.0 \pm 1.8) \text{ MeV},$$

$$f_{B_s^*} = (215.2 \pm 3.0) \text{ MeV},$$

where the uncertainties quoted above are merely those brought about by the dependence on the scale μ .

4 Cursory summary of observations, conclusions, and outlook

1. For beauty mesons, a strong correlation between m_b and the sum-rule result for f_B was observed:

$$\frac{\delta f_B}{f_B} \approx -8 \frac{\delta m_b}{m_b}.$$

Combining our sum-rule analysis with the latest results for f_B and f_{B_s} from lattice QCD implies

$$m_b = (4.247 \pm 0.027_{\text{OPE}} \pm 0.011_{\text{syst}} \pm 0.018_{\text{exp}}) \text{ GeV}.$$

2. Whereas the decay constants of charmed mesons, f_D , f_{D_s} , f_{D^*} , and $f_{D_s^*}$, obtained from QCD sum rules [11–13] turned out to be practically independent of the particular choice of the scale μ , in the beauty sector the situation is different: the decay constants of bottom mesons, particularly of vector bottom mesons, are very sensitive to the precise value of the scale μ . Averaging over the range $3 < \mu \text{ (GeV)} < 6$, we get the non-strange and strange bottom-meson decay-constant ratios

$$\frac{f_{B^*}}{f_B} = 0.923 \pm 0.059, \quad \frac{f_{B_s^*}}{f_{B_s}} = 0.932 \pm 0.047;$$

here, the above uncertainties incorporate the OPE-related uncertainties as well as the systematic uncertainties estimated along the course of our algorithm. If, however, one relies on calculations performed for low μ scales, $2.5 < \mu \text{ (GeV)} < 3.5$, then, to a good accuracy, one finds $f_{B^*}/f_B \approx 1$.

The unpleasant dependence of the QCD sum-rule predictions for the beauty-meson decay constants on the scale μ requires further detailed study in order to acquire better control over the related uncertainty.

References

- [1] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **147**, 385 (1979).
- [2] K. G. Chetyrkin and M. Steinhauser, Phys. Lett. B **502**, 104 (2001), arXiv:hep-ph/0012002.
- [3] K. G. Chetyrkin and M. Steinhauser, Eur. Phys. J. C **21**, 319 (2001), arXiv:hep-ph/0108017.
- [4] M. Jamin and B. O. Lange, Phys. Rev. D **65**, 056005 (2002), arXiv:hep-ph/0108135.
- [5] K. A. Olive *et al.* (Particle Data Group), Chin. Phys. C **38**, 090001 (2014).
- [6] W. Lucha, D. Melikhov, and S. Simula, Phys. Rev. D **79**, 096011 (2009), arXiv:0902.4202 [hep-ph].
- [7] W. Lucha, D. Melikhov, and S. Simula, J. Phys. G **37**, 035003 (2010), arXiv:0905.0963 [hep-ph].
- [8] W. Lucha, D. Melikhov, and S. Simula, Phys. Lett. B **687**, 48 (2010), arXiv: 0912.5017 [hep-ph].
- [9] W. Lucha, D. I. Melikhov, and S. Simula, Phys. Atom. Nucl. **73**, 1770 (2010), arXiv:1003.1463 [hep-ph].
- [10] W. Lucha, D. Melikhov, and S. Simula, Phys. Rev. D **88**, 056011 (2013), arXiv:1305.7099 [hep-ph].
- [11] W. Lucha, D. Melikhov, and S. Simula, J. Phys. G **38**, 105002 (2011), arXiv:1008.2698 [hep-ph].
- [12] W. Lucha, D. Melikhov, and S. Simula, Phys. Lett. B **701**, 82 (2011), arXiv:1101.5986 [hep-ph].
- [13] W. Lucha, D. Melikhov, and S. Simula, Phys. Lett. B **735**, 12 (2014), arXiv:1404.0293 [hep-ph].